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*Research Article*

# On Approximate $C^*$-Ternary $m$-Homomorphisms: A Fixed Point Approach

**M. Eshaghi Gordji,[1,2] Z. Alizadeh,[1,2] Y. J. Cho,[3] and H. Khodaei[1,2]**

[1] *Department of Mathematics, Semnan University, P.O. Box 35195-363, Semnan, Iran*
[2] *Center of Excellence in Nonlinear Analysis and Applications (CENAA), Semnan University, Semnan, Iran*
[3] *Department of Mathematics Education and the RINS, Gyeongsang National University, Chinju 660-701, Republic of Korea*

Correspondence should be addressed to Y. J. Cho, yjcho@gnu.ac.kr

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Using fixed point methods, we prove the stability and superstability of $C^*$-ternary additive, quadratic, cubic, and quartic homomorphisms in $C^*$-ternary rings for the functional equation $f(2x+y)+f(2x-y)+(m-1)(m-2)(m-3)f(y)=2^{m-2}[f(x+y)+f(x-y)+6f(x)]$, for each $m=1,2,3,4$.

## 1. Introduction

Following the terminology of [1], a nonempty set $G$ with a ternary operation $[\cdot,\cdot,\cdot]:G\times G\times G\to G$ is called a *ternary groupoid*, which is denoted by $(G,[\cdot,\cdot,\cdot])$. The ternary groupoid $(G,[\cdot,\cdot,\cdot])$ is said to be *commutative* if $[x_1,x_2,x_3]=[x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}]$ for all $x_1,x_2,x_3\in G$ and all permutations $\sigma$ of $\{1,2,3\}$. If a binary operation $o$ is defined on $G$ such that $[x,y,z]=(x\circ y)\circ oz$ for all $x,y,z\in G$, then we say that $[\cdot,\cdot,\cdot]$ is derived from $\circ$. We say that $(G,[\cdot,\cdot,\cdot])$ is a *ternary semigroup* if the operation $[\cdot,\cdot,\cdot]$ is associative, that is, if $[[x,y,z],u,v]=[x,[y,z,u],v]=[x,y,[z,u,v]]$ holds for all $x,y,z,u,v\in G$ (see [2]). Since it is extensively discussed in [3], the full description of a physical system $\mathbb{S}$ implies the knowledge of three basis ingredients: the set of the observables, the set of the states, and the dynamics that describes the time evolution of the system by means of the time dependence of the expectation value of a given observable on a given statue. Originally, the set of the observable was considered to be a $C^*$-algebra [4]. In many applications, however, it was shown not to be the most convenient choice and the $C^*$-algebra was replaced by a von

Neumann algebra because the role of the representation turns out to be crucial mainly when long-range interactions are involved (see [5] and references therein). Here we used a different algebraic structure.

A $C^*$*-ternary ring* is a complex Banach space $A$, equipped with a ternary product $(x,y,z) \to [x,y,z]$ of $A^3$ into $A$, which is $C$-linear in the outer variables, conjugate $C$-linear in the middle variable and associative in the sense that $[x,y,[z,w,v]] = [x,[w,z,y],v] = [[x,y,z],w,v]$ and satisfies $\|[x,y,z]\| \le \|x\| \cdot \|y\| \cdot \|z\|$ and $\|[x,y,z]\| = \|x\|^3$.

If a $C^*$-ternary ring $(A,[\cdot,\cdot,\cdot])$ has an identity, that is, an element $e \in A$ such that $x = [x,e,e] = [e,e,x]$ for all $x \in A$, then it is routine to verify that $A$, endowed with $x \circ y := [x,e,y]$ and $x^* := [e,x,e]$, is a unital $C^*$-algebra. Conversely, if $(A,\circ)$ is a unital $C^*$-algebra, then $[x,y,z] := x \circ y^* \circ z$ makes $A$ into a $C^*$-ternary algebra.

Consider the functional equation $\mathfrak{I}_1(f) = \mathfrak{I}_2(f)(\mathfrak{I})$ in a certain general setting. A function $g$ is an approximate solution of $(\mathfrak{I})$ if $\mathfrak{I}_1(g)$ and $\mathfrak{I}_2(g)$ are close in some sense. The Ulam stability problem asks whether or not there exists a true solution of $(\mathfrak{I})$ near $g$. A functional equation is said to be *superstable* if every approximate solution of the equation is an exact solution of the functional equation. The problem of stability of functional equations originated from a question of Ulam [6] concerning the stability of group homomorphisms.

Let $(G_1,*)$ be a group and $(G_2,\star,d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that, if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x * y), h(x) \star h(y)) < \delta \tag{1.1}$$

for all $x,y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x),H(x)) < \epsilon$ for all $x \in G_1$?

If the answer is affirmative, we say that the equation of homomorphism $H(x * y) = H(x) \star H(y)$ is *stable*. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

In 1941, Hyers [7] gave a first affirmative answer to the question of Ulam for Banach spaces.

Let $X$ and $Y$ be Banach spaces. Assume that $f : X \to Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon \tag{1.2}$$

for all $x,y \in X$ and some $\epsilon > 0$. Then there exists a unique additive mapping $T : X \to Y$ such that $\|f(x) - T(x)\| \le \epsilon$ for all $x \in X$.

A generalized version of the theorem of Hyers for approximately additive mappings was given by Aoki [8] in 1950 (see also [9]). In 1978, a generalized solution for approximately linear mappings was given by Th. M. Rassias [10]. He considered a mapping $f : X \to Y$ satisfying the condition

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.3}$$

for all $x,y \in X$, where $\epsilon \ge 0$ and $0 \le p < 1$. This result was later extended to all $p \ne 1$ and generalized by Gajda [11], Th. M. Rassias and Šemrl [12], and Isac and Th. M. Rassias [13].

In 2000, Lee and Jun [14] have improved the stability problem for approximately additive mappings. The problem when $p = 1$ is not true. Counter examples for the corresponding assertion in the case $p = 1$ were constructed by Gadja [11], Th. M. Rassias and Šemrl [12].

On the other hand, J. M. Rassias [15–17] considered the Cauchy difference controlled by a product of different powers of norm. Furthermore, a generalization of Th. M. Rassias theorems was obtained by Găvruţa [18], who replaced

$$\epsilon\left(\|x\|^p + \|y\|^p\right) \tag{1.4}$$

and $\epsilon\|x\|^p\|y\|^p$ by a general control function $\varphi(x,y)$. In 1949 and 1951, Bourgin [19, 20] is the first mathematician dealing with stability of (ring) homomorphism $f(xy) = f(x)f(y)$. The topic of approximation of functional equations on Banach algebras was studied by a number of mathematicians (see [21–33]).

The functional equation:

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.5}$$

is related to a symmetric biadditive mapping [34, 35]. It is natural that this equation is called a *quadratic functional equation*. For more details about various results concerning such problems, the readers refer to [36–43].

In 2002, Jun and Kim [44] introduced the following cubic functional equation:

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{1.6}$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.6). Obviously, the mapping $f(x) = cx^3$ satisfies the functional equation (1.6), which is called the *cubic functional equation*. In 2005, Lee et al. [45] considered the following functional equation

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y). \tag{1.7}$$

It is easy to see that the mapping $f(x) = dx^4$ is a solution of the functional equation (1.7), which is called the *quartic functional equation*.

## 2. Preliminaries

In 2007, Park and Cui [46] investigated the generalized stability of a quadratic mapping $f : A \to B$, which is called a $C^*$*-ternary quadratic mapping* if $f$ is a quadratic mapping satisfies

$$f([x,y,z]) = [f(x), f(y), f(z)] \tag{2.1}$$

for all $x, y, z \in A$. Let $(A, [\cdot,\cdot,\cdot])$ be a $C^*$-ternary ring derived from a unital commutative $C^*$-algebra $A$ and let $f : A \to A$ satisfy $f(x) = x^2$ for all $x \in A$. It is easy to show that the mapping $f : A \to A$ is a $C^*$-ternary quadratic mapping.

Recently, in 2010, Bae and Park [47] investigated the following functional equations

$$f(2x+y)+f(2x-y)=2^{m-2}[f(x+y)+f(x-y)+6f(x)] \tag{2.2}$$

for each $m=1,2,3$, and

$$f(2x+y)+f(2x-y)+6f(y)=4[f(x+y)+f(x-y)+6f(x)] \tag{2.3}$$

and they have obtained the stability of the functional equations (2.2) and (2.3).

We can rewrite the functional equations (2.2) and (2.3) by

$$\begin{aligned} f(2x+y)+f(2x-y)&+(m-1)(m-2)(m-3)f(y) \\ &=2^{m-2}[f(x+y)+f(x-y)+6f(x)]. \end{aligned} \tag{2.4}$$

Obviously, the monomial $f(x)=ax^m$ $(x\in\mathbb{R})$ is a solution of the functional equation (2.4) for each $m=1,2,3,4$.

For $m=1,2$, Bae and Park [47, 48] showed that the functional equation (2.4) is equivalent to the additive equation and quadratic equation, respectively.

If $m=3$, the functional equation (2.4) is equivalent to the cubic equation [44]. Moreover, Lee et al. [45] solved the solution of the functional equation (2.4) for $m=4$.

In this paper, using the idea of Park and Cui [46], we study the further generalized stability of $C^*$-ternary additive, quadratic, cubic, and quartic mappings over $C^*$-ternary algebra via fixed point method for the functional equation (2.4). Moreover, we establish the superstability of this functional equation by suitable control functions.

*Definition 2.1.* Let $A$ and $B$ be two $C^*$-ternary algebras.

(1) A mapping $f:A\to B$ is called a *$C^*$-ternary additive homomorphism* (briefly, $C^*$-ternary 1-homomorphism) if $f$ is an additive mapping satisfying (2.1) for all $x,y,z\in A$.

(2) A mapping $f:A\to B$ is called a *$C^*$-ternary quadratic mapping* (briefly, $C^*$-ternary 2-homomorphism) if $f$ is a quadratic mapping satisfying (2.1) for all $x,y,z\in A$.

(3) A mapping $f:A\to B$ is called a *$C^*$-ternary cubic mapping* (briefly, $C^*$-ternary 3-homomorphism) if $f$ is a cubic mapping satisfying (2.1) for all $x,y,z\in A$.

(4) A mapping $f:A\to B$ is called a *$C^*$-ternary quartic homomorphism* (briefly, $C^*$-ternary 4-homomorphism) if $f$ is a quartic mapping satisfying (2.1) for all $x,y,z\in A$.

Now, we state the following notion of fixed point theorem. For the proof, refer to [49] (see also Chapter 5 in [50] and [51, 52]). In 2003, Radu [53] proposed a new method for obtaining the existence of exact solutions and error estimations, based on the fixed point alternative (see also [54–57]).

Let $(X,d)$ be a generalized metric space. We say that a mapping $T:X\to X$ satisfies a Lipschitz condition if there exists a constant $L\geq 0$ such that $d(Tx,Ty)\leq Ld(x,y)$ for all $x,y\in X$, where the number $L$ is called the Lipschitz constant. If the Lipschitz constant

$L$ is less than 1, then the mapping $T$ is called a *strictly contractive mapping*. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity.

The following theorem was proved by Diaz and Margolis [49] and Radu [53].

**Theorem 2.2.** *Suppose that $(\Omega, d)$ is a complete generalized metric space and $T : \Omega \to \Omega$ is a strictly contractive mapping with the Lipschitz constant $L$. Then, for any $x \in \Omega$, either*

$$d\left(T^m x, T^{m+1} x\right) = \infty, \quad \forall m \geq 0, \tag{2.5}$$

*or there exists a natural number $m_0$ such that*

(1) *$d(T^m x, T^{m+1} x) < \infty$ for all $m \geq m_0$;*

(2) *the sequence $\{T^m x\}$ is convergent to a fixed point $y^*$ of $T$;*

(3) *$y^*$ is the unique fixed point of $T$ in $\Lambda = \{y \in \Omega : d(T^{m_0} x, y) < \infty\}$;*

(4) *$d(y, y^*) \leq (1/(1-L)) d(y, Ty)$ for all $y \in \Lambda$.*

## 3. Approximation of $C^*$-Ternary $m$-Homomorphisms between $C^*$-Ternary Algebras

In this section, we investigate the generalized stability of $C^*$-ternary $m$-homomorphism between $C^*$-ternary algebras for the functional equation (2.4).

Throughout this section, we suppose that $X$ and $Y$ are two $C^*$-ternary algebras. For convenience, we use the following abbreviation: for any function $f : X \to Y$,

$$\begin{aligned} \Delta_m f(x, y) = {} & f(2x + y) + f(2x - y) + (m-1)(m-2)(m-3) f(y) \\ & - 2^{m-2}\left[f(x + y) + f(x - y) + 6 f(x)\right] \end{aligned} \tag{3.1}$$

for all $x, y \in X$.

From now on, let $m$ be a positive integer less than 5.

**Theorem 3.1.** *Let $f : X \to Y$ be a mapping for which there exist functions $\varphi_m : X \times X \to [0, \infty)$ and $\psi_m : X \times X \times X \to [0, \infty)$ such that*

$$\left\| \Delta_m f(x, y) \right\| \leq \varphi_m(x, y), \tag{3.2}$$

$$\left\| f([x, y, z]) - [f(x), f(y), f(z)] \right\| \leq \psi_m(x, y, z) \tag{3.3}$$

*for all $x, y, z \in X$. If there exists a constant $0 < L < 1$ such that*

$$\begin{aligned} \varphi_m\left(\frac{x}{2}, \frac{y}{2}\right) &\leq \frac{L}{2^m} \varphi_m(x, y), \\ \psi_m\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) &\leq \frac{L}{2^{3m}} \psi_m(x, y, z) \end{aligned} \tag{3.4}$$

*for all $x, y, z \in X$, then there exists a unique $C^*$-ternary $m$-homomorphism $\mathfrak{F} : X \to Y$ such that*

$$\|f(x) - \mathfrak{F}(x)\| \le \frac{L}{2^{m+1}(1-L)} \varphi_m(x, 0) \tag{3.5}$$

*for all $x \in X$.*

*Proof.* It follows from (3.4) that

$$\lim_{n\to\infty} 2^{mn} \varphi_m\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0, \tag{3.6}$$

$$\lim_{n\to\infty} 2^{3mn} \psi_m\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0 \tag{3.7}$$

for all $x, y, z \in X$. By (3.6), $\lim_{n\to\infty} 2^{mn}\varphi_m(0,0) = 0$ and so $\varphi_m(0,0) = 0$. Letting $x = y = 0$ in (3.2), we get $f(0) \le \varphi_m(0,0) = 0$ and so $f(0) = 0$.

Let $\Omega = \{g : g : X \to Y,\ g(0) = 0\}$. We introduce a generalized metric on $\Omega$ as follows:

$$d(g, h) = d_{\varphi_m}(g, h) = \inf\{K \in (0, \infty) : \|g(x) - h(x)\| \le K\varphi_m(x, 0),\ \forall x \in X\}. \tag{3.8}$$

It is easy to show that $(\Omega, d)$ is a generalized complete metric space [55].

Now, we consider the mapping $T : \Omega \to \Omega$ defined by $Tg(x) = 2^m g(x/2)$ for all $x \in X$ and $g \in \Omega$. Note that, for all $g, h \in \Omega$ and $x \in X$,

$$\begin{aligned} d(g, h) < K &\Longrightarrow \|g(x) - h(x)\| \le K\varphi_m(x, 0) \\ &\Longrightarrow \left\|2^m g\left(\frac{x}{2}\right) - 2^m h\left(\frac{x}{2}\right)\right\| \le 2^m K\varphi_m\left(\frac{x}{2}, 0\right) \\ &\Longrightarrow \left\|2^m g\left(\frac{x}{2}\right) - 2^m h\left(\frac{x}{2}\right)\right\| \le LK\varphi_m(x, 0) \\ &\Longrightarrow d(Tg, Th) \le LK. \end{aligned} \tag{3.9}$$

Hence we see that

$$d(Tg, Th) \le Ld(g, h) \tag{3.10}$$

for all $g, h \in \Omega$, that is, $T$ is a strictly self-mapping of $\Omega$ with the Lipschitz constant $L$. Putting $y = 0$ in (3.2), we have

$$\left\|2f(2x) - 2^{m+1} f(x)\right\| \le \varphi_m(x, 0) \tag{3.11}$$

for all $x \in X$ and so

$$\left\| f(x) - 2^m f\left(\frac{x}{2}\right) \right\| \le \frac{1}{2}\varphi_m\left(\frac{x}{2}, 0\right) \le \frac{L}{2^{m+1}}\varphi_m(x, 0) \tag{3.12}$$

for all $x \in X$, that is, $d(f, Tf) \le L/2^{m+1} < \infty$.

Now, from Theorem 2.2, it follows that there exists a fixed point $\mathfrak{F}$ of $T$ in $\Omega$ such that

$$\mathfrak{F}(x) = \lim_{n\to\infty} 2^{mn} f\left(\frac{x}{2^n}\right) \tag{3.13}$$

for all $x \in X$ since $\lim_{n\to\infty} d(T^n f, \mathfrak{F}) = 0$.

On the other hand, it follows from (3.2), (3.6), and (3.13) that

$$\|\Delta_m \mathfrak{F}(x, y)\| = \lim_{n\to\infty} 2^{mn} \left\| \Delta_m f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \le \lim_{n\to\infty} 2^{mn} \varphi_m\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \tag{3.14}$$

for all $x, y \in X$ and so $\Delta_m \mathfrak{F}(x, y) = 0$. By the result in [44, 45, 47], $\mathfrak{F}$ is $m$-mapping and so it follows from the definition of $\mathfrak{F}$, (3.3) and (3.7) that

$$\begin{aligned}\|\mathfrak{F}([x, y, z]) - [\mathfrak{F}(x), \mathfrak{F}(y), \mathfrak{F}(z)]\| &= \lim_{n\to\infty} 2^{3mn} \left\| f\left(\frac{[x, y, z]}{2^{3n}}\right) - \left[f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right), f\left(\frac{z}{2^n}\right)\right] \right\| \\ &\le \lim_{n\to\infty} 2^{3mn} \varphi_m\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\ &= 0\end{aligned} \tag{3.15}$$

for all $x, y, z \in X$ and so $\mathfrak{F}([x, y, z]) = [\mathfrak{F}(x), \mathfrak{F}(y), \mathfrak{F}(z)]$.

According to Theorem 2.2, since $\mathfrak{F}$ is the unique fixed point of $T$ in the set $\Lambda = \{g \in \Omega : d(f, g) < \infty\}$, $\mathfrak{F}$ is the unique mapping such that

$$\|f(x) - \mathfrak{F}(x)\| \le K\varphi_m(x, 0) \tag{3.16}$$

for all $x \in X$ and $K > 0$. Again, using Theorem 2.2, we have

$$d(f, \mathfrak{F}) \le \frac{1}{1-L} d(f, Tf) \le \frac{L}{2^{m+1}(1-L)} \tag{3.17}$$

and so

$$\|f(x) - \mathfrak{F}(x)\| \le \frac{L}{2^{m+1}(1-L)}\varphi_m(x, 0) \tag{3.18}$$

for all $x \in X$. This completes the proof. □

**Corollary 3.2.** *Let $\theta, r, p$ be nonnegative real numbers with $r, p > m$ and $(3p - r)/2 \geq m$. Suppose that $f : X \to Y$ is a mapping such that*

$$\|\Delta_m f(x, y)\| \leq \theta(\|x\|^r + \|y\|^r), \tag{3.19}$$

$$\|f([x, y, z]) - [f(x), f(y), f(z)]\| \leq \theta(\|x\|^p \cdot \|y\|^p \cdot \|z\|^p) \tag{3.20}$$

*for all $x, y, z \in X$. Then there exists a unique $C^*$-ternary $m$-homomorphism $\mathfrak{F} : X \to Y$ satisfying*

$$\|f(x) - \mathfrak{F}(x)\| \leq \frac{\theta}{2(2^r - 2^m)}\|x\|^r \tag{3.21}$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 3.1 by taking

$$\varphi_m(x, y) := \theta(\|x\|^r + \|y\|^r), \qquad \psi_m(x, y, z) := \theta(\|x\|^p \cdot \|y\|^p \cdot \|z\|^p) \tag{3.22}$$

for all $x, y, z \in X$. Then we can choose $L = 2^{m-r}$ and so the desired conclusion follows. □

*Remark 3.3.* Let $f : X \to Y$ be a mapping with $f(0) = 0$ such that there exist functions $\varphi_m : X \times X \to [0, \infty)$ and $\psi_m : X \times X \times X \to [0, \infty)$ satisfying (3.2) and (3.3). Let $0 < L < 1$ be a constant such that

$$\varphi_m(2x, 2y) \leq 2^m L\varphi_m(x, y), \qquad \psi_m(2x, 2y, 2z) \leq 2^{3m} L\psi_m(x, y, z) \tag{3.23}$$

for all $x, y, z \in X$. By the similar method as in the proof of Theorem 3.1, one can show that there exists a unique $C^*$-ternary $m$-homomorphism $\mathfrak{F} : X \to Y$ satisfying

$$\|f(x) - \mathfrak{F}(x)\| \leq \frac{1}{2^{m+1}(1 - L)}\varphi_m(x, 0) \tag{3.24}$$

for all $x \in X$. For the case

$$\varphi_m(x, y) := \delta + \theta(\|x\|^r + \|y\|^r), \qquad \psi_m(x, y, z) := \delta + \theta(\|x\|^p \cdot \|y\|^p \cdot \|z\|^p), \tag{3.25}$$

where $\theta$, $\delta$ are nonnegative real numbers and $0 < r, p < m$ and $(3p - r)/2 \leq m$, there exists a unique $C^*$-ternary $m$-homomorphism $\mathfrak{F} : X \to Y$ satisfying

$$\|f(x) - \mathfrak{F}(x)\| \leq \frac{\delta}{2(2^m - 2^r)} + \frac{\theta}{2(2^m - 2^r)}\|x\|^r \tag{3.26}$$

for all $x \in X$.

In the following, we formulate and prove a theorem in superstability of $C^*$-ternary $m$-homomorphism in $C^*$-ternary rings for the functional equation (2.4).

**Theorem 3.4.** *Suppose that there exist functions* $\varphi_m : X \times X \to [0,\infty)$, $\psi_m : X \times X \times X \to [0,\infty)$ *and a constant* $0 < L < 1$ *such that*

$$\varphi_m\left(0, \frac{y}{2}\right) \leq \frac{L}{2^m}\varphi_m(0,y),$$
$$\psi_m\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{2^{3m}}\psi_m(x,y,z) \tag{3.27}$$

*for all* $x, y, z \in X$. *Moreover, if* $f : X \to Y$ *is a mapping such that*

$$\|\Delta_m f(x,y)\| \leq \varphi_m(0,y), \tag{3.28}$$

$$\|f([x,y,z]) - [f(x), f(y), f(z)]\| \leq \psi_m(x,y,z) \tag{3.29}$$

*for all* $x, y, z \in X$, *then* $f$ *is a* $C^*$*-ternary* $m$*-homomorphism.*

*Proof.* It follows from (3.27) that

$$\lim_{n\to\infty} 2^{mn}\varphi_m\left(0, \frac{y}{2^n}\right) = 0, \tag{3.30}$$

$$\lim_{n\to\infty} 2^{3mn}\psi_m\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0 \tag{3.31}$$

for all $x, y, z \in X$. We have $f(0) = 0$ since $\varphi_m(0,0) = 0$. Letting $y = 0$ in (3.28), we get $f(2x) = 2^m f(x)$ for all $x \in X$. By using induction, we obtain

$$f(2^n x) = 2^{mn} f(x) \tag{3.32}$$

for all $x \in X$ and $n \in \mathbb{N}$ and so

$$f(x) = 2^{mn} f\left(\frac{x}{2^n}\right) \tag{3.33}$$

for all $x \in X$ and $n \in \mathbb{N}$. It follows from (3.29) and (3.33) that

$$\begin{aligned}
&\|f([x,y,z]) - [f(x), f(y), f(z)]\| \\
&\quad = 2^{3mn}\left\| f\left(\frac{[x,y,z]}{2^{3n}}\right) - \left[f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right), f\left(\frac{z}{2^n}\right)\right]\right\| \\
&\quad \leq 2^{3mn}\psi_m\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)
\end{aligned} \tag{3.34}$$

for all $x, y, z \in X$, and $n \in \mathbb{N}$. Hence, letting $n \to \infty$ in (3.34) and using (3.31), we have $f([x,y,z]) = [f(x), f(y), f(z)]$ for all $x, y, z \in X$.

On the other hand, we have

$$\|\Delta_m f(x,y)\| = 2^{mn}\left\|\Delta_m f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| \leq 2^{mn}\varphi_m\left(0, \frac{y}{2^n}\right) \tag{3.35}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Thus, letting $n \to \infty$ in (3.35) and using (3.30), we have $\Delta_m f(x, y) = 0$ for all $x, y \in X$. Therefore, $f$ is a $C^*$-ternary $m$-homomorphism. This completes the proof. □

**Corollary 3.5.** *Let $\theta$, $r$, $s$ be nonnegative real numbers with $r > m$ and $s > 3m$. If $f : X \to Y$ is a function such that*

$$\left\| \Delta_m f(x, y) \right\| \leq \theta \left\| y \right\|^r, \qquad \left\| f([x, y, z]) - [f(x), f(y), f(z)] \right\| \leq \theta \left( \|x\|^s + \left\| y \right\|^s + \|z\|^s \right) \tag{3.36}$$

*for all $x, y, z \in X$, then $f$ is a $C^*$-ternary $m$-homomorphism.*

*Remark 3.6.* Let $\theta, r$ be nonnegative real numbers with $r < m$. Suppose that there exists a function $\varphi_m : X \times X \times X \to [0, \infty)$ and a constant $0 < L < 1$ such that

$$\varphi_m(2x, 2y, 2z) \leq 2^{3m} L \varphi_m(x, y, z) \tag{3.37}$$

for all $x, y, z \in X$. Moreover, if $f : X \to Y$ is a mapping such that

$$\left\| \Delta_m f(x, y) \right\| \leq \theta \left\| y \right\|^r, \qquad \left\| f([x, y, z]) - [f(x), f(y), f(z)] \right\| \leq \varphi_m(x, y, z) \tag{3.38}$$

for all $x, y, z \in X$, then $f$ is a $C^*$-ternary $m$-homomorphism.

In the rest of this section, assume that $X$ is a unital $C^*$-ternary algebra with the unit $e$ and $Y$ is a $C^*$-ternary algebra with the unit $e'$.

**Theorem 3.7.** *Let $\theta, r, p$ be positive real numbers with $r > m, p > m$ and $(3p-r)/2 \geq m$ (resp. $(3p-r)/2 \leq m$). Suppose that $f : X \to Y$ is a mapping satisfying (3.19) and (3.20). If there exist a real number $\lambda > 1$ and $x_0 \in X$ such that $\lim_{n\to\infty} \lambda^{mn} f(x_0/\lambda^n) = e'$ (resp. $\lim_{n\to\infty}(1/\lambda^{mn}) f(\lambda^n x_0) = e'$), then the mapping $f : X \to Y$ is a $C^*$-ternary $m$-homomorphism.*

*Proof.* By Corollary 3.2, there exists a unique $C^*$-ternary $m$-homomorphism $\mathfrak{F} : X \to Y$ such that

$$\left\| f(x) - \mathfrak{F}(x) \right\| \leq \frac{\theta}{2(2^r - 2^m)} \|x\|^r \tag{3.39}$$

for all $x \in X$. It follows from (3.39) that

$$\mathfrak{F}(x) = \lim_{n\to\infty} \lambda^{mn} f\left(\frac{x}{\lambda^n}\right) \quad \left(\mathfrak{F}(x) = \lim_{n\to\infty} \frac{1}{\lambda^{mn}} f(\lambda^n x)\right) \tag{3.40}$$

for all $x \in X$ and $\lambda > 1$. Therefore, by the assumption, we get that $\mathfrak{F}(x_0) = e'$.

Let $\lambda > 1$ and $\lim_{n\to\infty}\lambda^{mn} f(x_0/\lambda^n) = e'$. It follows from (3.20) that

$$\begin{aligned}
&\left\| [\mathfrak{F}(x), \mathfrak{F}(y), \mathfrak{F}(z)] - [\mathfrak{F}(x), \mathfrak{F}(y), f(z)] \right\| \\
&\quad = \left\| \mathfrak{F}([x, y, z]) - [\mathfrak{F}(x), \mathfrak{F}(y), f(z)] \right\| \\
&\quad = \lim_{n\to\infty} \lambda^{2mn} \left\| f\left(\left[\frac{x}{\lambda^n}, \frac{y}{\lambda^n}, z\right]\right) - \left[f\left(\frac{x}{\lambda^n}\right), f\left(\frac{y}{\lambda^n}\right), f(z)\right] \right\| \\
&\quad \le \theta \lim_{n\to\infty} \lambda^{2mn} \left[\frac{1}{\lambda^{2np}} \left(\|x\|^p \cdot \|y\|^p\right) \cdot \|z\|^p\right] \\
&\quad = 0
\end{aligned} \tag{3.41}$$

for all $x, y, z \in X$ and so $\mathfrak{F}([x, y, z]) = [\mathfrak{F}(x), \mathfrak{F}(y), f(z)]$ for all $x, y, z \in X$. Letting $x = y = x_0$ in the last equality, we get $f(z) = \mathfrak{F}(z)$ for all $z \in X$.

Similarly, one can show that $f(z) = \mathfrak{F}(z)$ for all $z \in X$ when $\lambda > 1$ and $\lim_{n\to\infty}(1/\lambda^{mn}) f(\lambda^n x_0) = e'$. Therefore, the mapping $f : X \to Y$ is a $C^*$-ternary $m$-homomorphism. This completes the proof. □

**Theorem 3.8.** *Let $\theta, r, p$ be positive real numbers with $r > m$ and $p > 2m$ and $(3p - r)/2 \ge m$ (resp. $(3p - r)/2 \le m$). Suppose that $f : X \to Y$ is a mapping satisfying* (3.19) *and*

$$\left\| f([x, y, z]) - [f(x), f(y), f(z)] \right\| \le \theta\left(\|x\|^p \cdot \|y\|^p + \|y\|^p \cdot \|z\|^p + \|x\|^p \cdot \|z\|^p\right) \tag{3.42}$$

*for all $x, y, z \in X$. If there exist a real number $\lambda > 1$ and $x_0 \in X$ such that $\lim_{n\to\infty}\lambda^{mn} f(x_0/\lambda^n) = e'$ (resp. $\lim_{n\to\infty}(1/\lambda^{mn}) f(\lambda^n x_0) = e'$), then the mapping $f : X \to Y$ is a $C^*$-ternary $m$-homomorphism.*

*Proof.* By Theorem 3.1 there exists a unique $C^*$-ternary $m$-homomorphism $\mathfrak{F} : X \to Y$ such that

$$\left\| f(x) - \mathfrak{F}(x) \right\| \le \frac{\theta}{2(2^r - 2^m)} \|x\|^r \tag{3.43}$$

for all $x \in X$. It follows from (3.43) that

$$\mathfrak{F}(x) = \lim_{n\to\infty} \lambda^{mn} f\left(\frac{x}{\lambda^n}\right) \qquad \left(\mathfrak{F}(x) = \lim_{n\to\infty} \frac{1}{\lambda^{mn}} f(\lambda^n x)\right) \tag{3.44}$$

for all $x \in X$ and $\lambda > 1$. Therefore, by the assumption, we get that $\mathfrak{F}(x_0) = e'$.

Let $\lambda > 1$ and $\lim_{n\to\infty}\lambda^{mn} f(x_0/\lambda^n) = e'$. It follows from (3.20) that

$$\begin{aligned}
&\left\| [\mathfrak{F}(x), \mathfrak{F}(y), \mathfrak{F}(z)] - [\mathfrak{F}(x), \mathfrak{F}(y), f(z)] \right\| \\
&\quad = \left\| \mathfrak{F}([x, y, z]) - [\mathfrak{F}(x), \mathfrak{F}(y), f(z)] \right\| \\
&\quad = \lim_{n\to\infty} \lambda^{2mn} \left\| f\left(\left[\frac{x}{\lambda^n}, \frac{y}{\lambda^n}, z\right]\right) - \left[f\left(\frac{x}{\lambda^n}\right), f\left(\frac{y}{\lambda^n}\right), f(z)\right] \right\| \\
&\quad \le \theta \lim_{n\to\infty} \lambda^{2mn} \left[\frac{1}{\lambda^{2np}} \|x\|^p \cdot \|y\|^p + \frac{1}{\lambda^{np}} \|y\|^p \cdot \|z\|^p + \frac{1}{\lambda^{np}} \|x\|^p \cdot \|z\|^p\right] \\
&\quad = 0
\end{aligned} \tag{3.45}$$

for all $x, y, z \in X$ and so $\mathfrak{F}([x, y, z]) = [\mathfrak{F}(x), \mathfrak{F}(y), f(z)]$ for all $x, y, z \in X$. Letting $x = y = x_0$ in the last equality, we get $f(z) = \mathfrak{F}(z)$ for all $z \in X$.

Similarly, one can show that $f(z) = \mathfrak{F}(z)$ for all $z \in X$ when $\lambda > 1$ and $\lim_{n\to\infty}(1/\lambda^{mn}) f(\lambda^n x) = e'$. Therefore, the mapping $f : X \to Y$ is a $C^*$-ternary $m$-homomorphism. This completes the proof. □

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